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# Operator Product Expansions in Four-dimensional Superconformal Field Theories

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## Abstract

The operator product expansion in four-dimensional superconformal field theory is discussed. It is demonstrated that the OPE takes a particularly simple form for certain classes of operators. These are chiral operators, principally of interest in theories with  $N = 1$  or  $N = 2$  supersymmetry, and analytic operators, of interest in  $N = 2$  and  $N = 4$ . It is argued that the Green's functions of such operators can be determined up to constants.

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A central rôle in two-dimensional conformal field theories is played by operator product expansions [1]. Indeed, all the properties of these theories can be encoded in their OPE's. The OPE of two primary fields yields not only primary fields, but also descendants of primary fields. However, strong constraints on the OPE follow from demanding that it be compatible with conformal symmetry. This procedure determines the space-time dependent coefficients with which the primary fields occur up to constants of proportionality. These constants, however, can only be determined by the details of the model, in effect what null states it possess. Given these constants, the rest of the OPE, that is all the dependence on the descendants, is determined by conformal symmetry. In particular, in minimal models, the OPE's close with only a finite number of primary fields, so that the correlation functions of such theories can in principle be calculated using the OPE and the two and three-point functions.

In four dimensions conformal field theories were studied in some detail in a general setting some time ago [2], but at that time no non-trivial examples were known. However, it is now known that there are many supersymmetric gauge theories which are superconformal, certainly in perturbation theory and perhaps beyond. These theories are: the maximally supersymmetric ( $N = 4$ ) models [3], a class of  $N = 2$  [4] models and certain  $N = 1$  theories [5]. It has further been conjectured that more supersymmetric theories may have non-trivial fixed points [6]. It is therefore appropriate to reconsider four-dimensional operator product expansions in the supersymmetric context.

It has been observed that the chiral sector of superconformally invariant theories in four dimensions has certain similarities with such sectors in two dimensional superconformal theories. In particular, the chiral and dilation weights of chiral operators are related if the theory is at a fixed point [7]. It has also been pointed out that the so-called analytic sectors of  $N = 2$  and  $N = 4$  theories seem to have similar properties [8]. Moreover, it has been argued that, although these supersymmetric theories are only invariant under a finite dimensional superconformal group, their very special form allows one to solve, non-perturbatively, for large classes of their Green's functions. In particular, one can determine the Green's functions in any chiral or anti-chiral sector and it is likely that one can also do this in the analytic sector [8]. In this paper, we give the operator product expansions in these sectors and find close similarities to the corresponding two dimensional results.

We begin by giving a discussion of operator product expansions and their conformal properties which applies in a general setting. For simplicity we shall take spacetime or superspace to be complex. Let us denote the complex, finite-dimensional (super)conformal group by  $G$ , for example, for four-dimensional  $N$ -extended supersymmetry  $G = SL(4|N)$ . The (super)conformal theories of interest to us are described by (super)fields which live on the (super)space  $P \backslash G$  where  $P$  is a parabolic subgroup of  $G$ . Which subgroups one should take in four dimensions can be found in reference [9, 8]. We will denote the coordinates of  $P \backslash G$  by  $X$ .

We define primary fields to be fields that transform under an induced representation of  $G$ . To keep life simple, we shall suppose that the fields are one-dimensional, i.e. transform under a one-dimensional subgroup of  $P$ . In many cases such fields are the most interesting

to consider. For an infinitesimal (super)conformal transformation we have

$$\delta\phi = V\phi + q\Delta\phi \quad (1)$$

where  $q$  is a charge associated with the representation (related to the dilation weight of the field) and  $V$  is the vector field generating the transformation on the coset space,  $V(X) = \delta X \frac{\partial}{\partial X}$ ,  $\delta X$  being the change in  $X$ . The function  $\Delta$  is a function that characterises the induced representation. We can choose coordinates such that the components of  $V$  are polynomials of degree 2 in the components of  $X$  and such that  $\Delta$  is a polynomial of degree 1. Descendants are space-time or superspace derivatives of primary fields. These will not in general transform as induced representations. Indeed, under certain conformal transformations descendants mix into primary fields. One can also take descendants to be given by group generators acting on the primary fields, but this is equivalent to the above description.

Now consider a complete set of operators  $\{\Phi_I\}$  comprising both primary fields  $\{\phi_i\}$  and their descendants. We shall assume that we can write an operator product expansion in the standard form,

$$\Phi_I(X_1)\Phi_J(X_2) = \sum_K f_{IJ}^K(X_1, X_2)\Phi_K(X_2) . \quad (2)$$

Applying an infinitesimal (super)conformal transformation to this and considering only primary fields on the left-hand side we find

$$\sum_K [(V_1 + V_2 + q_i\Delta_1 + q_j\Delta_2)f_{ij}^K]\Phi_K = \sum_K f_{ij}^K(\delta\Phi_K(2) - V_2\Phi_K(2)) , \quad (3)$$

where the subscripts 1 and 2 refer to the two points involved. Under a transformation for which  $\Delta$  is  $X$ -independent, the terms proportional to the primary fields do not have any contributions from the transformations of the descendants and we get

$$(V_1 + V_2 + (q_i + q_j - q_k)\Delta)f_{ij}^k(1, 2) = 0 . \quad (4)$$

Hence, for these transformations, the coefficients  $f_{ij}^k$  behave as a two point function with a total  $q$  weight of  $(q_i + q_j - q_k)$ . For the ordinary conformal group these transformations are translations, dilations and Lorentz rotations. However, these transformations are sufficient to determine  $f_{ij}^k$  up to a constant. In particular, if the fields under consideration are Lorentz scalars and the primary fields have dilation weights  $d_i (= q_i$  in this case) then

$$f_{ij}^k = \frac{c_{ij}^k}{[(x_1 - x_2)^2]^{\frac{1}{2}(d_i + d_j - d_k)}} , \quad (5)$$

where  $c_{ij}^k$  are constants and  $x_1$  and  $x_2$  are the positions of the primary fields  $\phi_i$  and  $\phi_j$  in spacetime.

For the remaining transformations the function  $\Delta$  is (super) space dependent. However, we can still examine only the primary field terms provided we take into account of transformations of descendants which result in primary fields. This calculation determines the coefficients of the lowest level descendants.

Let us illustrate the procedure for the simplest situation, i.e. the ordinary conformal group in four dimensions. The only remaining transformations are the special conformal transformations with parameter  $C_{\dot{\beta}\beta}$  for which  $\Delta = x^{\beta\dot{\beta}}C_{\dot{\beta}\beta}$ . Under this transformation only the lowest descendants, i.e. the set of fields  $\{\partial_{\alpha\dot{\alpha}}\phi_i\}$ , transform into primary fields. Including these terms explicitly, the OPE of equation (3) becomes

$$\phi_i(x_1)\phi_j(x_2) = \sum_k f_{ij}^k(x_1, x_2)\phi_k(x_2) + \sum_k f_{ij}^{k;\alpha\dot{\alpha}}(x_1, x_2)\partial_{\alpha\dot{\alpha}}\phi_k(x_2) + \dots \quad (6)$$

where the dots denote contributions from higher order descendants. Applying our previous argument, again except for special conformal transformations, we recover (5) and find that

$$f_{ij}^{k;\alpha\dot{\alpha}}(x_1, x_2) = \frac{(x_{12})^{\alpha\dot{\alpha}}}{(x_{12}^2)^{\frac{1}{2}(d_i+d_j-d_k)}} c_{ij}^{k(1)} , \quad (7)$$

where the  $c_{ij}^{k(1)}$  are constants and  $x_{12}^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} - x_2^{\alpha\dot{\alpha}}$ . Applying a special conformal transformation we find

$$c_{ij}^{k(1)} = c_{ij}^k \frac{(d_i - d_j + d_k)}{2d_k} . \quad (8)$$

By carrying out all conformal transformations on each side of the OPE and comparing coefficients of the descendant fields we can determine all the descendant contributions in terms of the constants  $c_{ij}^k$ . Thus the situation is the same as in two dimensional conformal field theories.

We now apply the above procedure to chiral superfields in four-dimensional  $N$ -extended supersymmetry. The analysis can be adapted straightforwardly to other dimensions where chiral fields are available. Due to the chiral constraint, a chiral superfield can be viewed as a function of only  $x^{\alpha\dot{\alpha}}$  and  $\theta^{\alpha a}$ , where  $x$  is an appropriate chiral variable and  $a = 1, \dots, N$ . The operator product expansion of two chiral superfields  $\phi_i(X_1)$  and  $\phi_j(X_2)$  can be written as

$$\begin{aligned} \phi_i(X_1)\phi_j(X_2) = & \sum_k \{ f_{ij}^k(X_1, X_2)\phi_k(X_2) + f_{ij}^{k;\alpha a}(X_1, X_2)(\partial_{\alpha a}\phi_k)(X_2) + \\ & + f_{ij}^{k;\alpha\dot{\alpha}}(X_1, X_2)(\partial_{\alpha\dot{\alpha}}\phi_k)(X_2) + \dots \} , \end{aligned} \quad (9)$$

where again the dots denote contributions from higher order descendants and where we have used the shorthand notation  $\partial_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$  and  $\partial_{\alpha a} = \frac{\partial}{\partial \theta^{\alpha a}}$ .

We now give the superconformal transformations written in terms of the chiral variables. The vector fields which generate the translations ( $P$ ), dilations ( $D$ ) and special conformal transformations ( $K$ ) are:

$$\begin{aligned} V(P)_{\alpha\dot{\alpha}} &= \partial_{\alpha\dot{\alpha}} \\ V(D) &= x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} \\ V(K)^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\beta}}x^{\beta\dot{\alpha}}\partial_{\beta\dot{\beta}} + x^{\beta\dot{\alpha}}\theta^{\alpha a}\partial_{\beta a} , \end{aligned} \quad (10)$$

and the associated  $\Delta$ 's are

$$\begin{aligned} \Delta(P)_{\alpha\dot{\alpha}} &= 0 \\ \Delta(D) &= 1 \\ \Delta(K)^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\alpha}} . \end{aligned} \quad (11)$$

The vector field generating internal symmetry ( $I$ ) transformations ( $SL(N)$ ) is

$$V(I)_a{}^b = \theta^{\alpha b} \partial_{\alpha a} - \frac{1}{N} \delta_a^b \theta^{\alpha c} \partial_{\alpha c} \quad (12)$$

and the function  $\Delta(I)_a^b$  vanishes in this case. For  $N \neq 4$  we also have  $R$ -symmetry transformations generated by

$$V(R) = \theta^{\alpha a} \partial_{\alpha a} \quad (13)$$

with

$$\Delta(R) = \frac{2N}{(N-4)} . \quad (14)$$

The  $Q$ -supersymmetry transformations are generated by

$$\begin{aligned} V(Q)_{\alpha a} &= \partial_{\alpha a} \\ V(Q)_{\dot{\alpha}}^a &= -\theta^{\alpha a} \partial_{\alpha \dot{\alpha}} , \end{aligned} \quad (15)$$

and the  $S$ -supersymmetry generators are

$$\begin{aligned} V(S)_a^{\dot{\alpha}} &= x^{\alpha \dot{\alpha}} \partial_{\alpha a} \\ V(S)^{\alpha a} &= -x^{\alpha \dot{\beta}} \theta^{\beta a} \partial_{\beta \dot{\beta}} + \theta^{\alpha b} \theta^{\beta a} \partial_{\beta b} , \end{aligned} \quad (16)$$

and only the last of these has a non-vanishing  $\Delta$  given by

$$\Delta(S)^{\alpha a} = -\theta^{\alpha a} . \quad (17)$$

There are also Lorentz transformations which act in the obvious way on the vector and spinor coordinates.

The transformations for which  $\Delta$  is constant can, according to our general arguments, be used to determine the superspace dependence of the coefficients of the primary chiral superfields. Translations and supersymmetry transformations imply that the coefficients are functions of  $x_{12}^{\alpha \dot{\alpha}} \equiv x_1^{\alpha \dot{\alpha}} - x_2^{\alpha \dot{\alpha}}$  and  $\theta_{12}^{\alpha a} = \theta_1^{\alpha a} - \theta_2^{\alpha a}$ .  $R$  symmetry implies that if  $f_{ij}^k$  is to be non-zero it must be proportional to  $\theta_{12}^{\alpha a}$  to the power  $q_i + q_j - q_k$ . (There are no chiral fields of interest in  $N = 4$  rigid supersymmetry, so this is always valid for the applications we have in mind.) Let us consider in detail the case when  $q_i + q_j = q_k$ . Dilation and Lorentz symmetry imply that

$$f_{ij}^k = c_{ij}^k, \quad f_{ij}^{k; \alpha a} = \theta^{\alpha a} c_{ij}^{k(2)} \quad (18)$$

and

$$f_{ij}^{k, \alpha \dot{\alpha}} = (x_{12})^{\alpha \dot{\alpha}} c_{ij}^{k(3)} . \quad (19)$$

To fix the descendant coefficients we use special conformal transformations and special ( $S$ ) supersymmetries. We find that the contribution given by  $\phi_k$  and its descendants to the OPE is

$$\begin{aligned} \phi_i(X_1) \phi_j(X_2) &= c_{ij}^k \left\{ \phi_k(X_2) + \frac{q_i}{q_k} \theta_{12}^{\alpha a} (\partial_{\alpha a} \phi_k)(X_2) + \frac{q_i}{q_k} x_{12}^{\alpha \dot{\alpha}} (\partial_{\alpha \dot{\alpha}} \phi_k)(X_2) \right\} \\ &\quad + \text{higher order descendants} . \end{aligned} \quad (20)$$

This result is essentially identical to the analogous result for two dimensional superconformal field theory.

We may also have contributions from primaries which are Lorentz scalars and which have  $q_i + q_j - q_k = 3$ , for  $N = 1$ , and  $q_i + q_j - q_k = 2$ , for  $N = 2$ . Such terms have leading contributions of the form

$$f_{ij}^k = c_{ij}^k \frac{\theta_{12}^2}{x_{12}^4} \quad (21)$$

for  $N = 1$ , and

$$f_{ij}^k = c_{ij}^k \frac{\theta_{12}^4}{x_{12}^4} \quad (22)$$

for  $N = 2$ . There are also primary fields with undotted spinor indices and internal indices. For example, in  $N = 1$ , one can have a contribution to the OPE of Lorentz scalars  $\phi_i$  and  $\phi_j$  from a spin one-half field  $\phi_{k\alpha}$  with charge  $q_k$  if  $q_i + q_j - q_k = \frac{3}{2}$ , and for which the leading contribution would be

$$f_{ij}^{k\alpha} = c_{ij}^k \frac{\theta_{12}^\alpha}{x_{12}^2} \quad (23)$$

However, for any pair of primary chiral fields, one always finds a finite number of primaries in the OPE on the right-hand side determined by the charges and spinorial representations involved.

We now consider harmonic superfields [10]. For the theories of most interest to us, i.e. the extended rigidly supersymmetric theories, superfields of this type occur in  $N = 4$  Yang-Mills theory and in the  $N = 2$  matter sector of  $N = 2$  theories. To be concrete we consider the former case but the formalism can be easily adapted to  $N = 2$ . The  $N = 4$  harmonic superspace of interest to us is the extension of Minkowski superspace by the internal space  $\mathbb{F} = S(U(2) \times U(2)) \backslash SU(4)$ , and the fields we wish to consider are analytic fields on this space, that is to say, fields which are analytic with respect to the internal space  $\mathbb{F}$ , and which are also Grassmann analytic ( $G$ -analytic). The latter means that they are annihilated by half of the superspace covariant derivatives, and therefore depend on only half of the odd coordinates, in a similar fashion to chiral fields. The difference is that the derivatives involve the coordinates of the internal space and this allows one to use a mixture of dotted and undotted spinor derivatives. These fields can be defined on a new superspace, analytic superspace, which is similar to chiral superspace. It has local coordinates

$$X = \{x^{\alpha\dot{\alpha}}, \lambda^{aa'}, \pi^{a\dot{\alpha}}, y^{aa'}\}$$

where  $a$  and  $a'$  can both take on two values. (Locally, the internal space is just like ordinary complex Minkowski space).

The operator product expansion for two analytic fields takes the form

$$\begin{aligned} \phi_i(X_1)\phi_j(X_2) = & \sum_k \{ f_{ij}^k(X_1, X_2)\phi_k(X_2) + f_{ij}^{k,\alpha\dot{\alpha}}(X_1, X_2)(\partial_{\alpha\dot{\alpha}}\phi_k)(X_2) \\ & + f_{ij}^{k,a\dot{\alpha}}(X_1, X_2)(\partial_{a\dot{\alpha}}\phi_k)(X_2) + f_{ij}^{k,\alpha a'}(X_1, X_2)(\partial_{\alpha a'}\phi_k)(X_2) \\ & + f_{ij}^{k,aa'}(X_1, X_2)(\partial_{aa'}\phi_k)(X_2) + \text{higher order descendants} \} . \end{aligned} \quad (24)$$

The superconformal transformations, when written in analytic coordinates, take a particularly simple form. The vector fields which generate them are, for translations, dilations,

Lorentz transformations ( $M$ ) and special conformal transformations,

$$\begin{aligned}
V(P)_{\alpha\dot{\alpha}} &= \partial_{\alpha\dot{\alpha}} \\
V(D) &= x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \frac{1}{2}\lambda^{\alpha a'}\partial_{\alpha a'} + \frac{1}{2}\pi^{a\dot{\alpha}}\partial_{a\dot{\alpha}} \\
V(M)_{\alpha}^{\beta} &= (x^{\alpha\dot{\gamma}}\partial_{\alpha\dot{\gamma}} + \lambda^{\alpha c'}\partial_{\alpha c'}) - \text{trace} \\
V(M)_{\dot{\alpha}}^{\dot{\beta}} &= (x^{\gamma\dot{\beta}}\partial_{\gamma\dot{\alpha}} + \pi^{c\dot{\beta}}\partial_{c\dot{\alpha}}) - \text{trace} \\
V(K)^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\beta}}x^{\beta\dot{\alpha}}\partial_{\beta\dot{\beta}} + x^{\beta\dot{\alpha}}\lambda^{\alpha b'}\partial_{\beta b'} + \pi^{b\dot{\alpha}}x^{\alpha\dot{\beta}}\partial_{b\dot{\beta}} + \pi^{b\dot{\alpha}}\lambda^{\alpha b'}\partial_{bb'} .
\end{aligned} \tag{25}$$

For internal symmetry transformations they are

$$\begin{aligned}
V(I)_{aa'} &= \partial_{aa'} \\
V(I) &= y^{aa'}\partial_{aa'} + \frac{1}{2}\lambda^{\alpha a'}\partial_{\alpha a'} + \frac{1}{2}\pi^{a\dot{\alpha}}\partial_{a\dot{\alpha}} \\
V(I)_a^b &= (\pi^{b\dot{\gamma}}\partial_{a\dot{\gamma}} + y^{bc'}\partial_{ac'}) - \text{trace} \\
V(I)_{a'}^{b'} &= (\lambda^{\gamma b'}\partial_{\gamma a'} + y^{cb'}\partial_{ca'}) - \text{trace} \\
V(I)^{aa'} &= y^{ab'}y^{ba'}\partial_{bb'} + \lambda^{\beta a'}y^{ab'}\partial_{\beta b'} + y^{ba'}\pi^{a\dot{\beta}}\partial_{b\dot{\beta}} + \lambda^{\alpha a'}\pi^{a\dot{\beta}}\partial_{\beta\dot{\beta}} .
\end{aligned} \tag{26}$$

For  $Q$ -supersymmetry transformations we have

$$\begin{aligned}
V(Q)_{\alpha a'} &= \partial_{\alpha a'} \\
V(Q)_{a\dot{\alpha}} &= \partial_{a\dot{\alpha}} \\
V(Q)_{\alpha}^a &= y^{ab'}\partial_{\alpha b'} + \pi^{a\dot{\alpha}}\partial_{\alpha\dot{\alpha}} \\
V(Q)_{\dot{\alpha}}^{a'} &= y^{ba'}\partial_{b\dot{\alpha}} - \lambda^{\beta a'}\partial_{\beta\dot{\alpha}} ,
\end{aligned} \tag{27}$$

while for  $S$ -supersymmetry we have

$$\begin{aligned}
V(S)_a^{\alpha} &= x^{\alpha\dot{\beta}}\partial_{a\dot{\beta}} + \lambda^{\alpha b'}\partial_{ab'} \\
V(S)_{a'}^{\dot{\alpha}} &= x^{\beta\dot{\alpha}}\partial_{\beta a'} - \pi^{b\dot{\alpha}}\partial_{ba'} \\
V(S)^{aa'} &= x^{\beta\dot{\alpha}}y^{ab'}\partial_{\beta b'} - \pi^{b\dot{\alpha}}y^{ab'}\partial_{bb'} + x^{\beta\dot{\alpha}}\pi^{a\dot{\beta}}\partial_{\beta\dot{\beta}} - \pi^{b\dot{\alpha}}\pi^{a\dot{\beta}}\partial_{b\dot{\beta}} \\
V(S)^{\alpha a'} &= y^{ba'}x^{\alpha\dot{\beta}}\partial_{b\dot{\beta}} + y^{ba'}\lambda^{\alpha b'}\partial_{bb'} - \lambda^{\beta a'}x^{\alpha\dot{\beta}}\partial_{\beta\dot{\beta}} - \lambda^{\beta a'}\lambda^{\alpha b'}\partial_{\beta b'} .
\end{aligned} \tag{28}$$

The non-zero  $\Delta$ 's are

$$\begin{aligned}
\Delta(K)^{\alpha\dot{\alpha}} &= x^{\alpha\dot{\alpha}} \\
\Delta(I) &= -1 \\
\Delta(I)^{aa'} &= -y^{aa'} \\
\Delta(S)^{a\dot{\alpha}} &= \pi^{a\dot{\alpha}} \\
\Delta(S)^{\alpha a'} &= -\lambda^{\alpha a'} .
\end{aligned} \tag{29}$$

The transformation for an analytic field with charge  $q$  takes the form given in equation (1); the charge is the charge of the field with respect to the internal  $U(1)$ , i.e. the  $U(1)$  of the isotropy group of the internal space  $\mathbb{F}$ . In harmonic superspace it would correspond to a field satisfying  $D_o\phi = q\phi$  where  $D_o$  is the derivative on  $SU(4)$  corresponding to this  $U(1)$ . In the case of  $N = 2$  there is also an  $R$ -symmetry transformation generated by

$$V(R) = \lambda^{\alpha}\partial_{\alpha} - \pi^{\dot{\alpha}}\partial_{\dot{\alpha}} . \tag{30}$$

(In  $N = 2$  analytic space the internal indices  $a$  and  $a'$  only take one value and so can be dropped.)

We shall now analyse the OPE for analytic fields using the method outlined above. One finds again that the primary coefficients  $\{f_{ij}^k\}$  must obey the same equations as a two-point function with total charge  $\frac{1}{2}(q_i + q_j - q_k)$ , i.e.,

$$(V_1 + V_2 + \frac{1}{2}(q_i + q_j - q_k)(\Delta_1 + \Delta_2))f_{ij}^k = 0 . \quad (31)$$

Now the basic two-point function in  $N = 4$  is the one for an Abelian Yang-Mills field strength tensor  $W$  which has charge  $q = 1$ . It is given by

$$< W(X_1)W(X_2) > \propto \frac{\hat{y}^2}{x^2} := g_{12} \quad (32)$$

where

$$\hat{y}^{aa'} = y^{aa'} + 2 \frac{\lambda^{\alpha a'} \pi^{a \dot{\alpha}} x_{\alpha \dot{\alpha}}}{x^2} . \quad (33)$$

One then has

$$f_{ij}^k = c_{ij}^k (g_{12})^{\frac{1}{2}(q_i + q_j - q_k)} . \quad (34)$$

for some constants  $\{c_{ij}^k\}$ .

We can determine the coefficients for the descendant fields in the same way as before and so we arrive at the result

$$\begin{aligned} \phi_i(X_1)\phi_j(X_2) &= \sum_{k=0} c_{ij}^k (g_{12})^{\frac{1}{2}(q_i + q_j - q_k)} \{ \phi_k(X_2) + \frac{1}{2}(q_i - q_j + q_k) \times \\ &\quad \times (x_{12}^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} + \lambda_{12}^{\alpha a'} \partial_{\alpha a'} + \pi_{12}^{a \dot{\alpha}} \partial_{a \dot{\alpha}} + y_{12}^{aa'} \partial_{aa'}) \phi_k(X_2) \} + \dots \end{aligned} \quad (35)$$

In an  $N = 4$  Yang-Mills theory with gauge group  $SU(M)$ , for example, the basic local analytic operators are given by the gauge-invariant powers of the field, i.e., they are the operators

$$A_m := \text{tr}(W)^m, \quad m = 2, \dots, (M-1) . \quad (36)$$

The operator  $A_m$  has charge  $q = m$ . In particular,  $A_2$  is the supercurrent which we shall denote by  $T$ . Its components include the energy-momentum tensor, the spacetime supersymmetry currents and the currents corresponding to the internal  $SU(4)$  symmetry group. The OPE for two  $T$ 's is

$$\begin{aligned} T(1)T(2) &= c_o (g_{12})^2 + c_2 g_{12} \{ T(2) + (x^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} + \lambda^{\alpha a'} \partial_{\alpha a'} + \pi^{a \dot{\alpha}} \partial_{a \dot{\alpha}} + y^{aa'} \partial_{aa'}) T(2) \} \\ &\quad + \text{finite terms} . \end{aligned} \quad (37)$$

For most four-dimensional theories the OPE of the energy-momentum tensor with itself does not close on itself. However, for  $N = 4$  Yang-Mills it does, and the result is strikingly similar to the two-dimensional case; indeed, we can rescale  $T$  such that  $c_2 = 1$  in which case it would be tempting to interpret  $c_o$  as a the central charge. We remark that it is only in  $N = 4$  that this can happen because in  $N = 1$  and  $N = 2$  the supercurrent is neither chiral nor analytic, and we believe that it is only these special types of superfields which have such simple OPE's. For a discussion of the OPE in  $N = 1$  supersymmetric theories we refer the reader to [11].



There may be operators other than the  $\{A_m\}$  and their descendants appearing in the analytic OPE. For example, given a gauge-invariant scalar superfield on super Minkowski space one can always construct a gauge-invariant analytic field on harmonic superspace by applying enough spinorial derivatives. A similar sort of situation can arise with chiral fields in  $N = 1$ . If  $\phi$  is chiral then so is  $\bar{D}^2\bar{\phi}$ . However, the latter is not in general primary unless  $\phi$  has weight  $\frac{1}{3}$ . Clearly, analytic operators obtained in this way will only be able contribute to the analytic OPE if they are primary. The lowest-dimensional analytic operator of this type that one can construct in  $N = 4$  has naïve dimension 6 so that, even if it is primary, it cannot contribute to the OPE of two supercurrents.

We conclude with a consequence of the OPE for analytic fields in either  $N = 2$  or  $N = 4$ . From a formal point of view the spacetime coordinate  $x$  and the internal coordinate  $y$  appear in a very symmetrical manner. Indeed, as we have remarked earlier, in  $N = 4$  the internal space is locally the same as (complex) Minkowski space. However, from a physical point of view spacetime and the internal space are completely different. In particular, the singularities which appear in the OPE as  $x_1$  approaches  $x_2$  are due to the usual difficulties encountered in defining local products of operators in quantum field theory. On the other hand, the rôle of  $y$  is simply to act as a device to help us exploit the internal symmetries of the theory. The internal space is compact, and no internal points need or should be removed from the domain of definition of Green's functions of many operators. Therefore singularities in the internal variables are completely spurious and must cancel. One way of seeing this is to note that any analytic operator can be reexpressed in terms of a polynomial in  $y$  with coefficients which are fields on ordinary super Minkowski space. If one examines the right-hand side of the analytic OPE above, one sees that the absence of singularities in  $y$  requires that  $\frac{1}{2}(q_i + q_j - q_k)$  be an integer, and furthermore that there can only be a finite number of primary fields occurring because otherwise one will introduce poles in  $y$  for sufficiently large values of  $q_k$ . Thus the situation is similar in some respects to that obtaining in two-dimensional minimal models. Given that the analytic OPE is valid, and that analyticity imposes finiteness of the number of primaries occurring in any given OPE, it is tempting to conclude that any Green's function of analytic operators can in principle be computed knowing the three-point functions and the OPE. Any such Green's function depends only on a few arbitrary constants, i.e. the  $\{c_{ij}^k\}$ 's and the constants in the three-point functions. In other words, the analytic OPE for  $N = 4$  (and for  $N = 2$ ) suggests that this sector of these theories is solvable in the full quantum theory. We note, however, that this result depends on some assumptions, principally the form of the OPE for analytic fields and the assumption that analyticity is maintained in the quantum theory. The latter seems to be natural given that the theories we are interested in are superconformal. In a future paper [12] we shall give a more detailed discussion of the Green's functions using analyticity and superconformal invariants.

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